Inverting Arrows and Derived Categories

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Definition

Let S be a collection of morphisms in a category $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to S

- is a category $S^{-1}\mathcal{C}$ with a functor $q : \mathcal{C} \longrightarrow S^{-1}\mathcal{C} \quad \ni$
  - $q(s)$ is an isomorphism for all $s \in S$.
  - Given any functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ such that $f(s)$ is an isomorphism for all $s \in S$, there is a unique functor $g : S^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ such that $gq \equiv f$ are naturally equivalent. Diagrammatically:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{q} & S^{-1}\mathcal{C} \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\, g} \\ & & \mathcal{D} \end{array}$$

commutes up to natural equivalence.

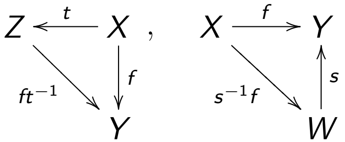
Existance, uniqueness and comments

- Existence is of $S^{-1}\mathcal{C}$ is not guaranteed.
- If exists, any two localizations are naturally equivalent.
- Obvious examples are our usual localization $S^{-1}A$ of commutative rings A along multiplicative sets S.
- **Remark.** The defintion of localization did not require that S is closed under isomorphisms.

Multiplicative System: Prelude

Suppose $f : X \longrightarrow Y, t : X \longrightarrow Z, s : W \longrightarrow Y$ be morphisms in $\mathcal{C}$, with $s, t \in S$. Required to define morphisms

$ft^{-1}, s^{-1}f \in S^{-1}\mathcal{C}$:



- **Question:** What would be the relationship between the collections $\{ft^{-1}\}, \{s^{-1}f\}$ of "left, right fractions"? Are they same, disjoint or none?
- Define "multiplicative systems" S, so that $\{ft^{-1}\} = \{s^{-1}f\}$ are same.

Definition

A collection S of morphisms in a category $\mathcal{C}$ is called a multiplicative system, if the following three (four) conditions hold:

- (**Closure**): If $s, t \in S$ are composable then $st \in S$.
- (**Ore Condition**): Suppose morphisms g, t be given, with $t \in S$. Then, there are morphisms s, f with $s \in S$ such that the diagram

$$\begin{array}{ccc} W & \overset{f}{\dashrightarrow} & Z \\ {\scriptstyle s}\downarrow{\scriptstyle \in S} & & {\scriptstyle t}\downarrow{\scriptstyle \in S} \\ X & \underset{g}{\longrightarrow} & Y \end{array} \qquad \textit{commute}$$

Continued

- (**Ore Condition**: Continued): *The symmetric condition also holds*: Suppose morphisms φ, ψ be given, with $\psi \in S$. Then, there are morphisms γ, θ with $\theta \in S$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \psi \downarrow {\color{red}\in S} & & \theta \downarrow {\color{red}\in S} \\ C & \xrightarrow[\gamma]{} & D \end{array} \qquad \textit{commute}$$

(*Ore condition ensures* $\{ft^{-1}\} = \{s^{-1}f\}$.)

Continued

- (**Cancellation**): For morphisms $f, g : X \longrightarrow Y$

  $$sf = sg \quad for\ some \quad s \in S \Longleftrightarrow ft = st \quad for\ some \quad t \in S.$$

  If $Mor_{\mathcal{C}}(*, *)$ have group structures then cancellation means

  $$sf = 0 \quad for\ some \quad s \in S \Longleftrightarrow ft = 0 \quad for\ some \quad t \in S.$$

- (**Identity**): For all objects $X \in \mathcal{C}$, the identity $1_X \in S$.

Left and Right Fractions

Suppose S is a multiplicative system in $\mathcal{C}$.

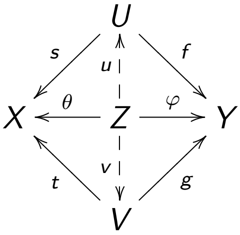
- By a left fraction fs^{-1} we mean a chain $\mathcal{C}$:

$$fs^{-1} : X \xleftarrow{s} Z \xrightarrow{f} Y \quad where \quad s \in S.$$

- Let $LF(\mathcal{C}, S)$ be the collection of all left fractions.

Equivalence of Fractions

Two left fractions $fs^{-1}, gt^{-1} : X \longrightarrow Y$ are defined to be equivalent, if there is another left fraction $\varphi\theta^{-1} : X \longrightarrow Y$ so that the diagram

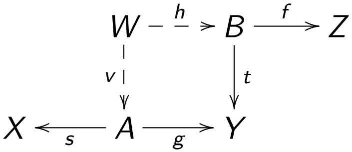


commutes for some u, v.

- This is an equivalence relation on $L\mathcal{F}(\mathcal{C}, S)$.

Composition of Fractions

Let $gs^{-1} : X \longrightarrow Y, ft^{-1} : Y \longrightarrow Z$ be two left fractions. Use Ore Condition and complete the commutative diagram:



Here $v \in S$.

Define composition $(ft^{-1})(gs^{-1}) := (fu)(sv)^{-1}$.

- Compostion is well defined upto equivalence.

The Category $S^{-1}\mathcal{C}$

Let S be a multiplicative system in a (small) category $\mathcal{C}$. Let $S^{-1}\mathcal{C}$ be the category defined as follows:

- The objects of $S^{-1}\mathcal{C}$ are same as the objects of $\mathcal{C}$.
- For objectes X, Y, let

$$Mor_{S^{-1}\mathcal{C}}(X, Y) := \textit{set of equivalence class of left fractions}$$

Existance Theorem of Gabriel-Zisman

- Define the "universal Functor" $q : \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$ as follows
  - For objects $X \in \mathcal{C}$, let $q(X) = X$.
  - For a morphism $f : X \longrightarrow Y \in \mathcal{C}$ define

$$q(f) = f(1_X^{-1}) : \ X \xrightarrow{\;1_X\;} X \xrightarrow{\;f\;} Y$$

  - **Theorem.** For a multiplicative system S in a category $\mathcal{C}$,localization $S^{-1}\mathcal{C}$ exists.
  - Namely, the functor q has the universal property of localization.

Categories of Modules

Let A be a noetherian commutative ring with $\dim A = d$. We use the notation $\mathcal{A}$ for the following categories.

- Let $\mathcal{A} = \mathcal{M}(A) = FGM(A)$ be the category of finitely generated $A-$modules.
- Let $\mathcal{A} = \mathcal{FL}(A)$ be the category of finitely generated $A-$modules with finite length.
- More generally, for inntegers $r \geq 0$, let be $\mathcal{A} = \mathcal{F}(A, r)$ the category of finitely generated $A-$modules M with

$$co\dim(Supp(M)) \geq k; \quad i.e. \qquad Height(Ann(M)) \geq r.$$

Abelian Categories

These "$\mathcal{A}$"s are abelian categories. This means

- $\mathcal{A}$ has a zero.
- $Hom(M, N)$ have abelian group structures.
- $\mathcal{A}$ is closed under finite product, kernel and cokernel.
- Every injective morphism is kernel of its cokernel.
- Every surjective morphism is cokernel of its kernel.

We will consider only the above abelian categories of modules.

The category $Ch(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category as above.

- Then, $Ch(\mathcal{A})$ will denote the category of chain complexes defined as follows:
  - The objects in $Ch(\mathcal{A})$ are the chain complexes of objects in $\mathcal{A}$).
  - For $A_\bullet, B_\bullet \in Ch(\mathcal{A})$, the morphisms $Hom_{Ch(\mathcal{A})}(A_\bullet, B_\bullet)$ is defined to be the group of chain complex maps.

The category $\mathbf{K}(\mathcal{A})$

The category $\mathbf{K}(\mathcal{A})$ is defined as follows:

- Objects of $\mathbf{K}(\mathcal{A})$ are same as that of $Ch(\mathcal{A})$.
- For $A_\bullet, B_\bullet \in \mathbf{K}(\mathcal{A})$, the morphisms are defined as

$$Hom_{\mathbf{K}(\mathcal{A})}(A_\bullet, B_\bullet) := \frac{Hom_{Ch(\mathcal{A})}(A_\bullet, B_\bullet)}{\sim}$$

where $\sim$ denotes the homotopy equivalence.

Subcategories of $Ch(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})$

- $Ch^b(\mathcal{A})$ will denote the "full subcategory" of $Ch(\mathcal{A})$ consisting of bounded complexes.
- Similarly, $\mathbf{K}^b(\mathcal{A})$ will denote the "full subcategory" of $\mathbf{K}(\mathcal{A})$ consisting of bounded complexes.
- Similar other such subcategories are defined.

The Derived Category $D(\mathcal{A})$

Definition (see [W])

Let $\mathcal{A}$ be a (small) abelian category, as above.

- A morphism $\varphi : P_\bullet \longrightarrow Q_\bullet \in K(\mathcal{A})$, is said to be a quasi-isomorphism, if $H^i(\varphi)$ is an isomorphism, $\forall\ i$.
- Let S be the set of all quasi-isomorphisms in $K(\mathcal{A})$. Then, S is a multiplicative system in $K(\mathcal{A})$.
- The Derived Category $D(\mathcal{A})$ is defined to be the localization $S^{-1}K(\mathcal{A})$.
- Likewise, we define the derived category $D^b(\mathcal{A})$ of bounded chain complexes is defined to be the localization $T^{-1}K^b(\mathcal{A})$, where T is set of all quasi-isomorphisms in $K^b(\mathcal{A})$.

Shift or Translation in $Ch(\mathcal{A})$

- Given an object $P_\bullet \in Ch(\mathcal{A})$, let $(P_\bullet[-1])_n = P_{n-1}$. Then, $P_\bullet[-1]$ is an object in $Ch(\mathcal{A})$.
- Also, define

$$T : Ch(\mathcal{A}) \longrightarrow Ch(\mathcal{A}) \quad \textit{by sending} \quad P_\bullet \mapsto P_\bullet[-1].$$

Then, T is an equivalence of categories.

Cone Constrution

Suppose $u : P_\bullet \longrightarrow Q_\bullet$ is a map of chain complexes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \xrightarrow{d_n^P} & P_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle u_{n+1}} & & \downarrow{\scriptstyle u_n} & & \downarrow{\scriptstyle u_{n-1}} & & \\ \cdots & \longrightarrow & Q_{n+1} & \longrightarrow & Q_n & \xrightarrow{d_n^Q} & Q_{n-1} & \longrightarrow & \cdots \end{array}$$

- Define $C_n = P_{n-1} \oplus Q_n$.
- Let $\partial_n : C_n \longrightarrow C_{n-1} = P_{n-2} \oplus Q_{n-1}$ be

$$\partial_n = \begin{pmatrix} -d_{n-1}^P & 0 \\ -u_{n-1} & d_n^Q \end{pmatrix}$$

Continued: Cone Constrution

- Then $C_\bullet$ is an object in $Ch(\mathcal{A})$ (i. e. a chain complex).
- $C_\bullet$ is called the cone of u; also denoted by $C_\bullet(u)$.
- For each n there is an exact sequence

$$0 \longrightarrow Q_n \longrightarrow C_n \longrightarrow P_{n-1} \longrightarrow 0$$

- This induces an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_\bullet & \longrightarrow & C_\bullet & \longrightarrow & P_\bullet[-1] & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & T(P_\bullet) & & \end{array}$$

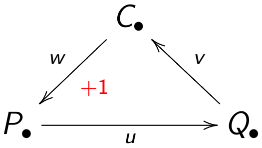
Triangles in $Ch(\mathcal{A})$

- The above leads to a long chain (not exact):

$$\cdots \longrightarrow P_\bullet \xrightarrow{u} Q_\bullet \longrightarrow C_\bullet(u)$$

$$\longrightarrow T(P_\bullet) \xrightarrow{T(u)} T(Q_\bullet) \longrightarrow T(C_\bullet(u)) \longrightarrow \cdots$$

- Such a chain is called (a jargon) a triangle and diagramatically represented as:



$$OR \quad P_\bullet \xrightarrow{u} Q_\bullet \xrightarrow{v} C_\bullet \xrightarrow{w} TP_\bullet$$

Triangulated Categories $K(\mathcal{A}), D(\mathcal{A})$

- We define and establish that $K(\mathcal{A}), D(\mathcal{A})$ are Triangulated Categories.
- In fact, the definition is abstraction of some of the properties of $K(\mathcal{A}), D(\mathcal{A})$.

The definition if Δed Categories

Definition

(see [W, 10.2.1]) A Triangulated Category **K** conisists of the following

- It is an additive category **K**.
- **K** is equipped with a natural equivalence $T : \mathbf{K} \xrightarrow{\sim} \mathbf{K}$, to be called the translation functor.
- **K** is also equipped with a distinguished family of triangles (u, v, w) of morphisms in **K**, to be called exact triangles.
- The translation and the exact triangles satisfies the following axioms (TR1-4).

Morphism of exact triangles

Before we give the axioms, we define morphisms of triangles.

- A morphism between two triangles $(u, v, w), (u', v', w')$, as in the diagram, is a triple (f, g, h) of morphisms in $\mathbf{K}$ such that

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Tf} \\ A' & \xrightarrow[u']{} & B' & \xrightarrow[v']{} & C' & \xrightarrow[w']{} & TA' \end{array} \quad \textit{commutes}.$$

Axiom TR1

- Every morphism $u : A \longrightarrow B$ in $\mathbf{K}$ can be embedded *inan exact triangle* (u, v, w) : $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$
- for all object $A \in \mathbf{K}$
$$A \xlongequal{1_A} A \xrightarrow{0} 0 \xrightarrow{0} TA \quad \textit{is an exact } \Delta.$$
- Given an isomorphim of two triangles:
$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \wr\downarrow f & & \wr\downarrow g & & \wr\downarrow h & & \wr\downarrow Tf \\ A' & \xrightarrow[u']{} & B' & \xrightarrow[v']{} & C' & \xrightarrow[w']{} & TA' \end{array}$$
if one line is exact, so is the other one.

Axiom TR2: (Rotation)

- Suppose

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA \quad \textit{is exact}.$$

Then, so are

$$B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB$$

and

$$T^{-1}C \xrightarrow{-T^{-1}w} A \xrightarrow{u} B \xrightarrow{v} C$$

Axiom TR3: (Morphisms)

Given two exact triangles, as in the diagram, and morphisms $f, g, \exists$ *a morphism* h,

$$\ni \quad \begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \downarrow f & & \downarrow g & & \exists \downarrow h & & \downarrow Tf \\ A' & \xrightarrow[u']{} & B' & \xrightarrow[v']{} & C' & \xrightarrow[w']{} & TA' \end{array} \quad \textit{commutes.}$$

Axiom TR4: (Octahedron)

The Octahedron axiom states how triangles over two morhisms u, v and the composition uv must interact.

- Suppose, two morphisms u, v are given. Consider triangels over $u, v, w := vu$ as in the diagram:

Continued: (Octahedron)

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{u_1} & U & \xrightarrow{u_2} & TX \\
\| & & \downarrow{\scriptstyle v} & & & & \| \\
X & \xrightarrow[w]{} & Z & \xrightarrow[w_1]{} & W & \xrightarrow[w_2]{} & TX \\
 & & \downarrow{\scriptstyle v_1} & & & & \\
 & & V & & & & \\
 & & \downarrow{\scriptstyle v_2} & & & & \\
 & & TY & & & &
\end{array}
$$

- By TR1, there is $f : U \longrightarrow W$ to fill in. The Contention is V will also sit below $f : U \longrightarrow W$.

Continued: (Octahedron)

Foramally, $\exists\ f, g$ that completes the following diagram:

$$
\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{u_1} & U & \xrightarrow{u_2} & TX \\
\| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f} & & \| \\
X & \xrightarrow{w} & Z & \xrightarrow{w_1} & W & \xrightarrow{w_2} & TX \\
 & & \downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle Tu} \\
 & & V & = & V & \xrightarrow{v_2} & TY \\
 & & \downarrow{\scriptstyle v_2} & & \downarrow{\scriptstyle (T(u_1)v_2} & & \\
 & & TY & \xrightarrow{Tu_1} & TU & &
\end{array}
\qquad \textit{commutes.}
$$

- The third vertical line is also exact.

$K(\mathcal{A})$ is Triangulated

Theorem

$K(\mathcal{A})$ is a Triangulated category.

- Given an object in $P_\bullet \in K(\mathcal{A})$, define translation $TP_\bullet := P_\bullet[-1]$.
- A triangle (u, v, w) is declaired exact, if it is isomorphic to the cone of a morphism $u : P_\bullet \longrightarrow Q_\bullet \in K(\mathcal{A})$. Diagramtically:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \downarrow \wr & & \downarrow \wr & & \wr \downarrow & & \downarrow \wr \\ P_\bullet & \xrightarrow[f]{} & Q_\bullet & \longrightarrow & C[f]_\bullet & \longrightarrow & TP_\bullet \end{array} \quad \textit{commutes.}$$

$D(\mathcal{A})$ is Triangulated

Theorem

$D(\mathcal{A})$ is a Triangulated category.

The triangulated structure is inherited from $K(\mathcal{A})$.

The Proof

- Only thing needs to tbe checked is that

$$A = A \longrightarrow 0 \longrightarrow TA \quad \text{is an } exact \quad \Delta.$$

- Let C be the cone of the identity map $1_A : A \xrightarrow{\sim} A$. Consider the diagram

$$\begin{array}{ccccccc} A & = & A & \longrightarrow & C & \longrightarrow & TA \\ \| & & \| & & \downarrow & & \| \\ A & = & A & \longrightarrow & 0 & \longrightarrow & TA \end{array}$$

- Now C is Homotopic to 0. (Routine Checking.)
- All the rectangles commute. (Routine Checking)

Derived Categories of Exact Categories

Suppose $\mathcal{E}$ is an exact category.

- The homotopy category $K(\mathcal{E})$ and the Derive category $D(\mathcal{E})$ are defined as above.
- Everything said above works for $K(\mathcal{E})$ and $D(\mathcal{E})$.
- Let $\mathcal{P}(A)$ be the category of finitely generated projective $A-$modules. Then, $\mathcal{P}(A)$ is an exact category.
- The Derived category $D(\mathcal{P}(A))$ is a triangulated category.

- Weibel, Charles A. *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp.
- PAUL BALMER, *Triangular Witt Groups Part I: The 12-Term Localization Exact Sequence,* K-Theory 19: 311?-63, 2000